ANSWER TO HOMEWORK III

Solution 1. (i) $u(t,x) = \frac{1}{6}t^3x$.

(ii) $u(t,x) = \begin{cases} \frac{1}{a^2}\left(-1+\cosh(at)\right)e^{ax}, & a \neq 0, \\ \frac{1}{2}t^2, & a = 0. \end{cases}$

(ii) $u(t,x) = t(1+x) + \cos x + \cos t\,(\sin x - \cos x)$.

Solution 2. (i) By direct computation,

$$
\begin{aligned}
&E(T) - E(0) \\
=&\frac{1}{2}\int_a^b \left(\partial_t u(t,x)\right)^2 - \left(\partial_t u(0,x)\right)^2 dx + \frac{1}{2}\int_a^b \left(\partial_x u(T,x)\right)^2 - \left(\partial_x u(0,x)\right)^2 dx \\
=&\int_a^b\int_0^T \partial_t u(t,x)\cdot\partial_x^2 u(t,x) + \partial_x u(t,x)\cdot\partial_x\partial_t u(t,x) dt dx \\
=&\int_0^T \partial_x u(b,x)\cdot\partial_t u(b,x) - \partial_x u(t,a)\cdot\partial_t u(t,a) dt \\
=&-\int_0^T \left(\partial_t u(t,b)\right)^2 dt.
\end{aligned}
$$

(ii) Firstly ,we show that $u_t + u_x = 0$ in the region $A = \{t \geq b - x\}$. For $(t_0, x_0) \in A$, let $z(s) = \partial_t u(t_0+s, x_0-s) + \partial_x u(t_0+s, x_0-s)$, then

$$\frac{dz(s)}{ds} = 0.$$

Since $z(x_0 - b) = 0$, therefore $z(0) = 0$ which implies

$$\partial_t u(t_0,x_0) + \partial_x u(t_0,x_0) = 0.$$

Secondly, we show that $u \equiv 0$ in the region $B = \{t \geq x+b-2a\}$. For $(t_0,x_0) \in B$, let $z(s) = u(t_0+s, x_0+s)$, then by the above result,

$$\frac{dz(s)}{ds} = 0.$$

Since $z(-x_0+a) = 0$, therefore $z(0) = 0$ which implies

$$u(t_0,x_0) = 0.$$

Finally, for $(t_0,x_0) \in \{t \geq 2(b-a)\} \subset B$, it is clear that $u \equiv 0$.

Solution 3. (i) By direct computation, for $\omega \in \partial B_1(0)$, and $u \in C_c^\infty(\mathbb{R}^3)$,

$$
\begin{aligned}
-2\int_0^\infty u(r\omega)\cdot\omega\cdot\nabla_x u(r\omega)\cdot r dr =& -2\int_0^\infty u(r\omega)\cdot\frac{du(r\omega)}{dr}\cdot r dr \\
=& -\int_0^\infty \frac{d}{dr}\left(|u(r\omega)|^2\right)\cdot r dr \\
=& \int_0^\infty |u(r\omega)|^2 dr.
\end{aligned}
$$

(ii) By the above result and Cauchy-Schwarz inequality,

$$\begin{aligned}\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} dx =& \int_0^\infty \int_{\partial B_1} |u(r\omega)|^2 dS_\omega dr \\ =& -2 \int_0^\infty \int_{\partial B_1} u(r\omega) \cdot \omega \cdot \nabla_x u(r\omega) \cdot r dS_\omega dr \\ \leq& 2 \left(\int_0^\infty \int_{\partial B_1} |u(rw)|^2 dS_\omega dr \right)^{\frac{1}{2}} \left(\int_0^\infty \int_{\partial B_1} |\nabla_x u(rw)|^2 r^2 dS_w dr \right)^{\frac{1}{2}} \\ \leq& 2 \left(\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} dx \right)^{\frac{1}{2}} \left(\int_{\mathbb{R}^3} |\nabla u(x)|^2 dx \right)^{\frac{1}{2}},\end{aligned}$$

which implies

$$\int_{\mathbb{R}^3} \frac{|u(x)|^2}{|x|^2} dx \leq 4 \int_{\mathbb{R}^3} |\nabla u(x)|^2 dx.$$

Solution 4. (i) Since

$$u(t,x) = \frac{1}{4\pi t^2} \int_{\partial B_t(x)} f(y) + \nabla f(y) \cdot (y-x) + t g(y) dS_y := I_1 + I_2 + I_3.$$

For I_1,

$$I_1 = \frac{1}{4\pi t^2} \int_{\partial B_1(0)} f(x+t\omega) \cdot t^2 dS_\omega = \frac{1}{4\pi} \int_{\partial B_1(0)} f(x+t\omega) dS_\omega = u_1.$$

For I_2,

$$I_2 = \frac{1}{4\pi t^2} \int_{\partial B_1(0)} \nabla_y f(x+t\omega) \cdot (t\omega) \cdot t^2 dS_\omega = \frac{t}{4\pi} \int_{\partial B_1(0)} \nabla_y f(x+t\omega) dS_\omega = u_3.$$

For I_3,

$$I_3 = \frac{1}{4\pi t^2} \int_{\partial B_1(0)} t g(x+t\omega) \cdot t^2 dS_\omega = \frac{t}{4\pi} \int_{\partial B_1(0)} g(x+t\omega) dS_\omega = u_2.$$

which implies the result.

(ii) By Cauchy-Schwarz inequality,

$$\begin{aligned}\int_0^\infty |u_1(t,x)|^2 dt \leq& C \int_0^\infty \int_{\partial B_1(0)} |f(x+t\omega)|^2 dS_\omega dt \\ \leq& C \int_{\mathbb{R}^3} \frac{|f(x+y)|^2}{|y|^2} dy.\end{aligned}$$

$$\begin{aligned}\int_0^\infty |u_2(t,x)|^2 dt \leq& C \int_0^\infty t^2 \int_{\partial B_1(0)} |g(x+t\omega)|^2 dS_\omega dt \\ \leq& C \int_{\mathbb{R}^3} |g(x+y)|^2 dy.\end{aligned}$$

$$\begin{aligned}\int_0^\infty |u_3(t,x)|^2 dt \leq& C \int_0^\infty t^2 \int_{\partial B_1(0)} |\nabla_y f(x+t\omega)|^2 dS_\omega dt \\ \leq& C \int_{\mathbb{R}^3} |\nabla f(x+y)|^2 dy.\end{aligned}$$

(iii) By the above results and Hardy's inequality,

$$\begin{aligned}\int_0^\infty |u(t,x)|^2 dt \leq & C\left(\int_{\mathbb{R}^3} \frac{|f(x+y)|^2}{|y|^2} dy + \int_{\mathbb{R}^3} |g(x+y)|^2 dy + \int_{\mathbb{R}^3} |\nabla f(x+y)|^2 dy\right) \\ \leq & C\left(\int_{\mathbb{R}^3} |\nabla f(x+y)|^2 dy + \int_{\mathbb{R}^3} |g(x+y)|^2 dy\right) \\ \leq & C\left(\int_{\mathbb{R}^3} |\nabla f(y)|^2 dy + \int_{\mathbb{R}^3} |g(y)|^2 dy\right).\end{aligned}$$